Towards Mirror Symmetry on Noncompact Calabi-Yau Manifolds

TETSUJI KIMURA
Theory Division, Institute of Particle and Nuclear Studies,
High Energy Accelerator Research Organization (KEK)
Tsukuba, Ibaraki 305-0801, Japan
E-mail: tetsuji@post.kek.jp

ABSTRACT

We study one class of linear sigma models and their T-dualized theories for noncompact Calabi-Yau manifolds. In the low energy limit, we find that this system has various massless effective theories with orbifolding symmetries. This phenomenon is new and there are no analogous structures in the models for compact Calabi-Yau manifolds and for line bundles on the simple toric varieties.

1. Introduction and Summary

We constructed metrics on noncompact Calabi-Yau (CY) manifolds in the framework of $\mathcal{N} = 2$ supersymmetric nonlinear sigma models [1]. These manifolds are canonical line bundles on compact Einstein-Kähler manifolds, whose mirror pairs have not been well-known yet. In order to understand the mirror pairs, we first study canonical line bundles on hypersurfaces of projective spaces, the proto-types of [1], and their mirror pairs in terms of the linear sigma model [2] and its T-dualized theory [3]. First we discuss the linear sigma model. If the Fayet-Iliopoulos (FI) parameter is positive, the linear sigma model reduces to a supersymmetric nonlinear sigma model whose target space is a canonical line bundle on the degree k hypersurface of $\mathbb{C}\mathbf{P}^{N-1}$, i.e., a noncompact CY manifold. When we set the FI parameter to be negative, there appear various kinds of orbifolded theories as massless effective theories. Next we study the low energy limit of the T-dualized theory of the linear sigma model. We obtain two kinds of Landau-Ginzburg (LG) theories with superpotentials including positive and negative powers of twisted chiral superfields. Furthermore we construct two kinds of orbifolded noncompact CY geometries, which are related to the above two LG theories.

2. Linear Sigma Model and its T-dualized Theory

Let us first consider the linear sigma model [2] whose field configuration and chiral superpotential are as follows:[a]

chiral superfield	S_1	$\cdots$	S_N	P_1	P_2
$U(1)$ charge	1	$\cdots$	1	$-k$	$-N+k$

$$\widetilde{W} = -\Sigma\, t\,, \qquad W_{\text{LSM}} = P_1 \cdot G_k(S_i)\,, \tag{1}$$

where S_i, P_1 and P_2 are charged chiral superfields and Σ is a twisted chiral superfield coming from the gauge multiplet; t is a complex parameter combined with the FI parameter r and the Theta angle θ such as $t = r - i\theta$; $G_k(S_i)$ is a quasi-homogeneous polynomial of degree k written by N chiral superfields S_i.[b] The quasi-homogeneity insists that when

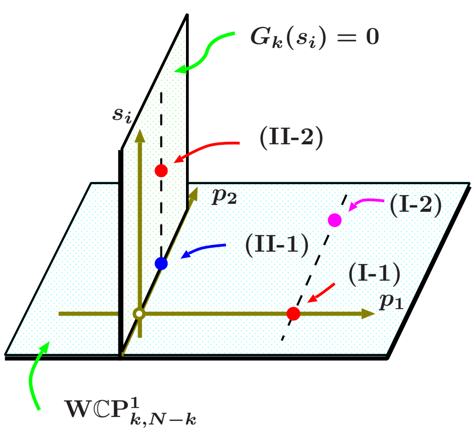
[a] We omit the explicit expression of the Lagrangian, see Ref. [2].
[b] For simplicity we set $2 \leq k \leq N-1$.

$G_k(S)$ and its derivatives $\partial_i G_k(S)$ vanish the variables S_i are all zero. The above field configuration (1) satisfies the "CY condition" such as $\sum_a Q_a = N - k - (N-k) = 0$. Due to this, the FI parameter does not receive one-loop renormalization effects and we find that this system is free from the UV divergence through perturbative calculations. The potential energy density for the scalar component fields is given by

$$\begin{aligned} U \ &= \ \frac{e^2}{2}\Big\{ r - \sum_{i=1}^N |s_i|^2 + k|p_1|^2 + (N-k)|p_2|^2 \Big\}^2 + \big|G_k(s_i)\big|^2 + |p_1|^2 \cdot \sum_{i=1}^N \big|\partial_i G_k(s_j)\big|^2 \\ &\quad +2|\sigma|^2 \Big(\sum_{i=1}^N |s_i|^2 + k^2 |p_1|^2 + (N-k)^2 |p_2|^2 \Big) \, . \end{aligned} \tag{2}$$

Let us analyze classical supersymmetric vacuum manifolds $U = 0$ and massless effective theories on them. When $r > 0$, we find that the vacuum manifold $\mathcal{M}_{\rm CY}$ is nothing but the canonical line bundle on the degree k hypersurface of projective space represented as $\mathcal{O}(-N+k) \to \mathbb{C}\mathbf{P}^{N-1}[k]$. In the IR limit $e \to \infty$, the linear sigma model reduces to the $\mathcal{N} = 2$ supersymmetric nonlinear sigma model on $\mathcal{M}_{\rm CY}$.

In $r < 0$, the shape of the vacuum manifold drastically changes to a very complicated one. Furthermore, there exist four massless effective theories on this vacuum space expressed in Figure 1, where $\mathbf{W}\mathbb{C}\mathbf{P}^1_{k,N-k}$, $\mathcal{M}_k^{p_1 \neq 0}$ and $\mathcal{M}_k^{p_1 = 0}$ are defined in (3):



(I-1) : at $\langle p_1 \rangle \neq 0$, $\langle p_2 \rangle = 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$$\left\{ \begin{array}{l} \textbf{(sigma model on } \mathcal{M}_k^{p_1 \neq 0}\textbf{)} \\ \otimes \textbf{ (LG theory with } W_{\rm LG} = \langle p_1 \rangle G_k(S)\textbf{)} \end{array} \right\} \Big/ \mathbb{Z}_k$$

(I-2) : at $\langle p_1 \rangle \neq 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$$\left\{ \begin{array}{l} \textbf{(sigma model on } \mathcal{M}_k^{p_1 \neq 0}\textbf{)} \\ \otimes \textbf{ (LG theory with } W_{\rm LG} = \langle p_1 \rangle G_k(S)\textbf{)} \end{array} \right\} \Big/ \mathbb{Z}_\alpha$$

$\alpha = \text{GCM}\{k, N-k\}$

(II-1) : at $\langle p_1 \rangle = 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle = 0$, massless DOF $= N+1$

$\{$**LG theory with** $W_{\rm LG} = P_1 \cdot G_k(S)\}/\mathbb{Z}_{N-k}$

(II-2) : at $\langle p_1 \rangle = 0$, $\langle p_2 \rangle \neq 0$, $\langle s_i \rangle \neq 0$, massless DOF $= N-1$

sigma model on $\mathcal{M}_k^{p_1 = 0}$

Figure 1: Four massless effective theories on the vacuum manifold, when $r < 0$.

$$\begin{aligned} \mathbf{W}\mathbb{C}\mathbf{P}^1_{k,N-k} \ &= \ \Big\{ (p_1, p_2) \in \mathbb{C}^2 \, \Big| \, r = -k|p_1|^2 - (N-k)|p_2|^2 \Big\} \, , \quad \mathcal{M}_k^{p_1 \neq 0} = \big\{ \mathbf{W}\mathbb{C}\mathbf{P}^1_{k,N-k} \, \big| \, p_1 \neq 0 \big\} \, , \\ \mathcal{M}_k^{p_1 = 0} \ &= \ \Big\{ (p_2, s_i) \in \mathbb{C}^* \times \mathbb{C}^N \, \Big| \, r - \sum_{i=1}^N |s_i|^2 + (N-k)|p_2|^2 = G_k(s_i) = 0 \Big\} \, . \end{aligned} \tag{3}$$

Notice the following comments: The LG superpotential $W_{\rm LG} = \langle p_1 \rangle G_k(S)$ has an isolated singularity at $S_i = 0$, but $W_{\rm LG} = P_1 \cdot G_k(S)$ has no isolated singularities. Thus there may be no descriptions for (II-1) *as a minimal model*. (There still exists a possibility that (II-1) could be described as another, but unknown yet, well-defined CFT.) The massless theories (I-2) and (II-2) are obtained by deformations of VEVs in (I-1) and (II-1), respectively.

When $k=2$, the massless theories (I-1) and (I-2) reduce to the orbifolded sigma models on $\mathcal{M}^{p_1\neq 0}$, because $W_{\rm LG} = \langle p_1 \rangle G_{k=2}(S)$ is quadratic and generates mass terms of S_i.

In the classical level, there are no selection rules whether the four (or intrinsically two) theories will be realized as the true massless effective theory. However we can consider the T-dualized theories of them even in the classical level.

Now let us briefly discuss the T-dualized theory of the linear sigma model.[c] Under T-duality, a chiral superfield Φ_a in the linear sigma model is replaced by a twisted chiral superfields Y_a such as $2\overline{\Phi}_a \, {\rm e}^{2Q_a V} \Phi_a = Y_a + \overline{Y}_a$. Note that the imaginary part of Y_a is periodic with period 2π. The T-dualized theory is described by an exact twisted superpotential $\widetilde{W}$ and the period integral $\widehat{\Pi}$ in terms of Y_a and Σ:

$$\widetilde{W} \ = \ \Sigma\Big(\sum_{i=1}^N Y_i - k Y_{P_1} - (N-k) Y_{P_2} - t \Big) + \sum_{i=1}^N {\rm e}^{-Y_i} + {\rm e}^{-Y_{P_1}} + {\rm e}^{-Y_{P_2}} \ , \tag{4a}$$

$$\widehat{\Pi} \ = \ \int {\rm d}\Sigma \prod_{i=1}^N {\rm d}Y_i \, {\rm d}Y_{P_1} \, {\rm d}Y_{P_2} \, (k\Sigma) \, \exp\big(- \widetilde{W} \big) \ . \tag{4b}$$

In the IR limit $e \to \infty$, the dynamics of Σ is frozen and thus the twisted superpotential reduces to $\widetilde{W} = \sum_a {\rm e}^{-Y_a}$ with a constraint $\sum_a Q_a Y_a - t = 0$. We always take this procedure when we analyze low energy effective theories. Note that there exists the variable $k\Sigma$ in (4b) due to the existence of the superpotential $W_{\rm LSM}$ in the linear sigma model.

Here we construct mirror LG theories by replacing $k\Sigma$ into $k\frac{\partial}{\partial t}$ and by performing field re-definitions. First we consider the mirror LG theory of the effective theory (I-1) in Figure 1. Re-defining $X_i \equiv {\rm e}^{-\frac{1}{k}Y_i}$ and $X_{P_2} \equiv {\rm e}^{\frac{N-k}{k}Y_{P_2}}$, we find that the measure in (4b) remains canonical and the twisted LG superpotential becomes

$$\Big\{ \widetilde{W}_k \ = \ X_1^k + \cdots + X_N^k + X_{P_2}^{-\frac{k}{N-k}} + {\rm e}^{\frac{t}{k}} X_1 \cdots X_N X_{P_2} \Big\} \Big/ (\mathbb{Z}_k)^N \ . \tag{5}$$

Note that the orbifold symmetry $(\mathbb{Z}_k)^N$, derived from $Y_a \to Y_a + 2\pi i$, acts on X_i and X_{P_2} such as $X_i \to \omega_i X_i$ and $X_{P_2} \to \omega_{P_2} X_{P_2}$, where ω_i and ω_{P_2} satisfy $\omega_i^k = \omega_{P_2}^{-\frac{k}{N-k}} = \omega_1 \cdots \omega_N \omega_{P_2} = 1$. The negative power term $X_{P_2}^{-\frac{k}{N-k}}$ emerges due to the existence of the noncompact direction in the original CY. The similar situation appears when we consider the LG theory of the deformed conifold [4] and so on.

We can also perform another field re-definition $X_i \equiv {\rm e}^{-\frac{1}{N-k}Y_i}$ and $X_{P_1} \equiv {\rm e}^{\frac{k}{N-k}Y_{P_1}}$ preserving a canonical measure in (4b). Due to this, we obtain another twisted LG superpotential

$$\Big\{ \widetilde{W}_{N-k} \ = \ X_1^{N-k} + \cdots + X_N^{N-k} + X_{P_1}^{-\frac{N-k}{k}} + {\rm e}^{\frac{t}{N-k}} X_1 \cdots X_N X_{P_1} \Big\} \Big/ (\mathbb{Z}_{N-k})^N \ . \tag{6}$$

Note that the orbifold symmetry $(\mathbb{Z}_{N-k})^N$ acts on X_i and X_{P_1} such as $X_i \to \omega_i X_i$ and $X_{P_1} \to \omega_{P_1} X_{P_1}$, where ω_i and ω_{P_1} satisfy $\omega_i^{N-k} = \omega_{P_1}^{-\frac{N-k}{k}} = \omega_1 \cdots \omega_N \omega_{P_1} = 1$. By virtue of this orbifold symmetry, we can conjecture that this twisted theory is a mirror LG of the low energy theory (II-1).

[c]The generic formulation of the T-dualized action are discussed in Ref. [3].

Replacing $k\Sigma$ into $\frac{\partial}{\partial Y_{P_1}}$ in (4b), we obtain two CY geometries. The first one is

$$\begin{aligned}\widetilde{\mathcal{M}}_k &= \Big\{(Z_i; u, v) \in \mathbb{C}^{N+2} \,\Big|\, \mathcal{F}(Z_i) = 0 \,,\ \mathcal{G}(Z_b; u, v) = 0\Big\} \Big/ \big\{\mathbb{C}^* \times (\mathbb{Z}_k)^{N-2}\big\} \,, \\ \mathcal{F}(Z_i) &= \sum_{a=1}^{k} Z_a^k + \psi Z_1 \cdots Z_k \,, \quad \mathcal{G}(Z_b; u, v) \ = \sum_{b=k+1}^{N} Z_b^k + 1 - uv \,, \quad \psi \ = \ \mathrm{e}^{t/k} Z_{k+1} \cdots Z_N \,, \end{aligned} \tag{7}$$

where u and v are complex fields related to the noncompact direction of the original CY $\mathcal{M}_{\mathrm{CY}}$; Z_i are related to Y_i and $(\mathbb{Z}_k)^{N-2}$ acts on Z_i; the $\mathbb{C}^*$ symmetry acts on the first k fields Z_a. We can read this geometry as a CY geometry, i.e., a $(\mathbb{Z}_k)^{N-2}$ orbifold CY hypersurface $\mathbb{C}\mathbf{P}^{k-1}[k]$ parametrized by ψ (this is represented by $\mathcal{F} = 0$) under the constraint $\mathcal{G} = 0$. Similarly we also obtain another CY geometry:

$$\begin{aligned}\widetilde{\mathcal{M}}_{N-k} &= \Big\{(Z_i; u, v) \in \mathbb{C}^{N+2} \,\Big|\, F(Z_a) = 0 \,,\ G(Z_i; u, v) = 0\Big\} \Big/ \big\{\mathbb{C}^* \times (\mathbb{Z}_{N-k})^{N-2}\big\} \,, \\ F(Z_a) &= \sum_{a=1}^{k} Z_a^{N-k} + 1 \,,\ G(Z_i; u, v) = \sum_{b=k+1}^{N} Z_b^{N-k} + \psi Z_{k+1} \cdots Z_N \,,\ \psi = \big(\mathrm{e}^{\frac{t}{N-k}} - uv\big) Z_1 \cdots Z_k \,. \end{aligned} \tag{8}$$

This is a CY hypersurface $\mathbb{C}\mathbf{P}^{N-k-1}[N-k]$ parametrized by ψ (this is described by $G = 0$) under the constraint $F = 0$ with orbifold symmetry $(\mathbb{Z}_{N-k})^{N-2}$.

3. Discussions

From the viewpoint of orbifold symmetries, it seems that the sigma models on (7) and the sigma model on (8) are mirror duals of the theories (I-1) and (II-1), respectively. But it is now very hard to check this conjecture and to determine the true mirror geometry of the original CY because we have not known how to define topological invariants on *noncompact* geometries. However, we might understand the true vacuum in the linear sigma model and evaluate the correct LG theory and CY geometry in the T-dualized theory when we evaluate free energies of the low energy effective theories in Figure 1. We will explain the above discussions more precisely, and try to solve them in [5].

4. Acknowledgements

The author would like to thank Hiroyuki Fuji, Takahiro Masuda, Shun'ya Mizoguchi, Kazutoshi Ohta, Hitoshi Sato and Dan Tomino for valuable comments. This work was supported in part by the JSPS Research Fellowships for Young Scientists (No.15-03926).

5. References

[1] K. Higashijima, T. Kimura and M. Nitta, *Ann. Phys.* **296**, 347 (2002), `hep-th/0110216`; *Nucl. Phys.* **B645**, 438 (2002), `hep-th/0202064`.
[2] E. Witten, *Nucl. Phys.* **B403**, 159 (1993), `hep-th/9301042`.
[3] K. Hori and C. Vafa, `hep-th/0002222`.
[4] D. Ghoshal and C. Vafa, *Nucl. Phys.* **B453**, 121 (1995), `hep-th/9506122`.
[5] T. Kimura, in preparation.